



Queen Mary University of London

B.Sc. EXAMINATION *By Course Unit*

MAS204 Calculus III

Friday, 2 May 2003
2:30pm-5:30pm

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each section.

Calculators are NOT permitted in this examination.

You are reminded of the following, which you may use without proof..

In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are

$$h_1 = 1, h_2 = r, h_3 = r \sin \theta.$$

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are

$$h_1 = 1, h_2 = r, h_3 = 1.$$

SECTION A

Each question carries 8 marks. You should attempt ALL questions.

1. (a) [5 Marks] Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ along the curve C on which $(x, y, z) = (t^{1/2}, t, t^{3/2})$, from $t = 0$ to $t = 1$.
(b) [3 Marks] Calculate ∇f for $f = xy^2z^3$ and evaluate it at $(3, 2, 1)$.
2. $\mathbf{F} = r\mathbf{e}_r + r \cos \theta \mathbf{e}_\theta + r \sin \theta \cos \phi \mathbf{e}_\phi$ in spherical polar coordinates. Calculate
(a) [3 Marks] $\nabla \cdot \mathbf{F}$
(b) [5 Marks] $\nabla \times \mathbf{F}$
3. Using index notation, or otherwise:
(a) [4 Marks] Prove that $\nabla(r^2 \mathbf{a} \cdot \mathbf{r}) = r^2 \mathbf{a} + 2\mathbf{r}(\mathbf{a} \cdot \mathbf{r})$, where $\mathbf{a}$ is a constant vector, $\mathbf{r}$ is the usual position vector with components (x, y, z) in Cartesian coordinates and $r^2 = x^2 + y^2 + z^2$.
(b) [4 Marks] Prove that $\nabla \times (\nabla \times r^2 \mathbf{a}) = -4\mathbf{a}$.
[You may assume the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.]
4. (a) [3 Marks] State the Divergence Theorem.
(b) [5 Marks] Using this theorem, or otherwise, evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = x^2yz\mathbf{i} + y^2xz\mathbf{j} + z^2xy\mathbf{k}$ over the surface S of the cube defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.
5. For each of the following two vector fields either find the general solution V of $\mathbf{F} = \nabla V$ or give a reason why no such V exists.
(a) [6 Marks] $\mathbf{F} = x\mathbf{i} + y^2\mathbf{j} + z^3\mathbf{k}$.
(b) [2 Marks] $\mathbf{F} = z \cos \theta \mathbf{e}_r + r z \mathbf{e}_\theta + r \sin \theta \mathbf{e}_z$, in cylindrical polar coordinates.
6. Show that at a general point the following are solutions of Laplace's equation $\nabla^2 V = 0$.
(a) [4 Marks] $V = (r^2 + z^2)^{-1/2}$, in cylindrical polar coordinates.
(b) [4 Marks] $V = (3 \cos^2 \theta - 1)r^{-3}$, in spherical polar coordinates.
7. Find the Fourier series of the function $f = (x + |x|)/2$ on the range $(-\pi, \pi)$.

SECTION B

Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

1. (a) [3 Marks] State Stokes's theorem carefully.
Evaluate $\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} dS$ for $\mathbf{A} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ and the surface S of a semi-sphere, $x^2 + y^2 + z^2 = 1$, $z \geq 0$,
- (b) [10 Marks] by direct integration (you may find it useful to recall that x , y and z expressed in spherical polar coordinates are $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$), taking the normal outward from the semi-sphere,
- (c) [7 Marks] by using Stokes's theorem.
- (d) [2 Marks] Check agreement with one aspect of Stokes's theorem by directly calculating $\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} dS$ over the disk $z = 0$, $x^2 + y^2 \leq 1$ and showing that with an appropriate direction for $\mathbf{n}$ it agrees with the previous answer.
2. The function $f(x) = x(\pi - x)$ is given in the range $0 < x < \pi$. Find for $f(x)$
- (a) [9 Marks] the Fourier sine series,
- (b) [9 Marks] the Fourier cosine series for $f(x)$.
- (c) [2 Marks] By adding the two series obtained, or otherwise, show that the Fourier series for the function $g(x)$ defined by $g = 0$ on $-\pi < x \leq 0$ and $g = f$ on $0 < x < \pi$ is

$$g(x) = \frac{\pi^2}{12} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{\cos 2kx}{k^2} + \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k+1)x}{2k+1}.$$

- (d) [2 Marks] Show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2} = \frac{\pi^2}{6}.$$

3. The most general axisymmetric solution of Laplace's equation $\nabla^2 V = 0$ in spherical polar coordinates is written in the following form

$$V(r, \theta) = \sum_{n=0}^{\infty} R_n(r) P_n(\cos \theta).$$

The $P_n(x)$ are the Legendre polynomials, which are solutions of Legendre's equation

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_n(x)}{dx} \right] + n(n+1)P_n(x) = 0,$$

normalized by convention so that $P_n(1) = 1$.

- (a) [4 Marks] Check that the first few Legendre polynomials $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ are indeed solutions of Legendre's equation.
- (b) [11 Marks] Using Laplace's equation in spherical polar coordinates and Legendre's equation, derive the equation for the functions R_n and show that

$$R_n = A_n r^n + B_n r^{-n-1},$$

where A_n and B_n are arbitrary constants for each n .

- (c) [7 Marks] Consider the region D defined by $1 \leq r \leq 2$. Find the solution V of Laplace's equation, independent of ϕ , which satisfies the following boundary conditions: i) $V = 3 \cos^2 \theta + \cos \theta - 1$, when $r = 1$ and ii) $V = 3 \cos \theta$, when $r = 2$.

4. (a) [8 Marks] Show that if $y(x)$ maximizes or minimizes the integral

$$I[y] = \int_{x_0}^{x_1} F(x, y') dx,$$

with $y(x_0)$ and $y(x_1)$ fixed and F given, then y satisfies

$$\frac{\partial F}{\partial y'} = \text{const.}$$

(a first integral of the Euler-Lagrange equation).

- (b) [14 Marks] A particle moves in (x, y) plane with a speed which depends on x only as $v(x) = v_0 x$ in the range $0 \leq x \leq 1$. Show that (assuming the particle travels on the path of shortest time) the trajectory of the particle between the points $(0, 0)$ and $(1, 1)$ in the (x, y) plane satisfies

$$y = 1 - \sqrt{1 - x^2}.$$